

# SEQUENTIALLY COHEN-MACAULAY MIXED PRODUCT IDEALS

GIANCARLO RINALDO

ABSTRACT. We classify the ideals of mixed products that are sequentially Cohen-Macaulay.

## 1. INTRODUCTION

The class of ideals of mixed products is a special class of square-free monomial ideals. They were first introduced by G. Restuccia and R. Villarreal (see [8] and [12]), who studied the normality of such ideals.

In [6], C. Ionescu and G. Rinaldo studied the Castelnuovo-Mumford regularity, the depth and dimension of mixed product ideals and characterize when they are Cohen-Macaulay. In [9] the author calculated the Betti numbers of their finite free resolutions. In [5], L. T. Hoa and N. D. Tam studied these ideals in a broader situation.

Let  $S = K[\mathbf{x}, \mathbf{y}]$  be a polynomial ring over a field  $K$  in two disjoint sets of variables  $\mathbf{x} = \{x_1, \dots, x_n\}$ ,  $\mathbf{y} = \{y_1, \dots, y_m\}$ . The *ideals of mixed products* are the proper ideals

$$(1.1) \quad \sum_{i=1}^s I_{q_i} J_{r_i} \quad q_i, r_i \in \mathbb{Z}_{\geq 0}$$

where  $I_{q_i}$  (resp.  $J_{r_i}$ ) is the ideal of  $S$  generated by all the square-free monomials of degree  $q_i$  (resp.  $r_i$ ) in the variables  $\mathbf{x}$  (resp.  $\mathbf{y}$ ). We set  $I_0 = J_0 = S$  and  $I_{q_i} = (0)$  (resp.  $J_{r_i} = (0)$ ) if  $q_i > n$  (resp.  $r_i > m$ ). In the articles mentioned only two summands of 1.1 are allowed. In this article we classify the ideals of mixed product that are sequentially Cohen-Macaulay and Cohen-Macaulay for any  $s \in \mathbb{N}$ . Recently, a number of authors have been interested in classifying sequentially Cohen-Macaulay rings related to combinatorial structures (for example see [3], [4], [11]). This paper is inserted in this area and the tools used are essentially Stanley-Reisner rings and Alexander dual.

In section 2 we recall some preliminaries about simplicial complexes and questions related to commutative algebra. In section 3 we study the primary decomposition of mixed product ideals, we introduce the vectors  $\bar{q}$ ,  $\bar{r}$  that

---

2000 *Mathematics Subject Classification.* Primary 13H10; Secondary 13P10.

*Key words and phrases.* Mixed product ideals, sequentially Cohen-Macaulay rings, simplicial complexes.

are uniquely determined by the values  $q_i, r_i$  for  $i = 1, \dots, s, m$  and  $n$ , and we classify the Cohen-Macaulay mixed product ideals in terms of the vectors  $\bar{q}$  and  $\bar{r}$ . The vectors  $\bar{q}$  and  $\bar{r}$  are used also to classify the sequentially Cohen-Macaulay mixed product ideals in the last section.

## 2. PRELIMINARIES

In this section we recall some concepts on simplicial complexes that we will use in the article (see [1], [7], [10]).

Set  $V = \{x_1, \dots, x_n\}$ . A *simplicial complex*  $\Delta$  on the vertex set  $V$  is a collection of subsets of  $V$  such that (i)  $\{x_i\} \in \Delta$  for all  $x_i \in V$  and (ii)  $F \in \Delta$  and  $G \subseteq F$  imply  $G \in \Delta$ . An element  $F \in \Delta$  is called a *face* of  $\Delta$ . For  $F \subset V$  we define the *dimension* of  $F$  by  $\dim F = |F| - 1$ , where  $|F|$  is the cardinality of the set  $F$ . A maximal face of  $\Delta$  with respect to inclusion is called a *facet* of  $\Delta$ . If all facets of  $\Delta$  have the same dimension, then  $\Delta$  is called *pure*.

A simplicial complex  $\Delta$  is called *shellable* if the facets of  $\Delta$  can be given a linear order  $F_1, \dots, F_t$  such that for all  $1 \leq i < j \leq t$ , there exist some  $v \in F_j \setminus F_i$  and some  $k \in \{1, \dots, j-1\}$  with  $F_j \setminus F_k = \{v\}$ .

Moreover, a pure simplicial complex  $\Delta$  is *strongly connected* if for every two facets  $F$  and  $G$  of  $\Delta$  there is a sequence of facets  $F = F_0, F_1, \dots, F_t = G$  such that  $\dim(F_i \cap F_{i+1}) = \dim \Delta - 1$  for each  $i = 0, \dots, t-1$ .

The *Stanley-Reisner ideal* of  $\Delta$ , denoted by  $I_\Delta$ , is the squarefree monomial ideal of  $S = K[x_1, \dots, x_n]$  generated by

$$\{x_{i_1}x_{i_2}\cdots x_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n, \{x_{i_1}, \dots, x_{i_p}\} \notin \Delta\},$$

and  $K[\Delta] = K[x_1, \dots, x_n]/I_\Delta$  is called the *Stanley-Reisner ring* of  $\Delta$ . It is known that

$$(2.1) \quad I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$$

with  $P_F = (\{x_1, \dots, x_n\} \setminus F)$ .

Let  $I = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_q}) \subset K[\mathbf{x}] = K[x_1, \dots, x_n]$  be a square-free monomial ideal, with  $\alpha_i = (\alpha_{i_1}, \dots, \alpha_{i_n}) \in \{0, 1\}^n$ . The *Alexander dual* of  $I$  is the ideal

$$(2.2) \quad I^* = \bigcap_{i=1}^q \mathfrak{m}_{\alpha_i},$$

where  $\mathfrak{m}_{\alpha_i} = (x_j : \alpha_{i_j} = 1)$ . It is known that  $(I^*)^* = I$ . We also have that if  $I, J$  are squarefree monomial ideals of  $S = K[x_1, \dots, x_n]$  then

$$(2.3) \quad (I + J)^* = I^* \cap J^*.$$

## 3. COHEN-MACAULAY MIXED PRODUCT IDEALS

Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  be a polynomial ring over a field  $K$  and let

$$(3.1) \quad \sum_{i=1}^s I_{q_i} J_{r_i}, \quad q_i, r_i \in \mathbb{Z}_{\geq 0}, \quad s \in \mathbb{N},$$

be an ideal of mixed product as in 1.1. In this section we study the primary decomposition of the ideal 3.1 and give a criterion for its Cohen-Macaulayness. Under the assumption that no summands in 3.1 is a subset of another summand, we set

$$(3.2) \quad 0 \leq q_1 < q_2 < \dots < q_s \leq n.$$

Under this assumption and because of the ordering 3.2 we have

$$(3.3) \quad 0 \leq r_s < r_{s-1} < \dots < r_1 \leq m.$$

Throughout this paper we always assume 3.2 and 3.3.

**Proposition 3.1.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ , then*

$$(3.4) \quad \left( \sum_{i=1}^s I_{q_i} J_{r_i} \right)^* = I_{n-q_1+1} + \sum_{i=1}^{s-1} I_{n-q_{i+1}+1} J_{m-r_i+1} + J_{m-r_s+1}.$$

*Proof.* We prove the assertion by induction on  $s$ . If  $s = 1$  we have that either  $q_1 = 0$  (resp.  $r_1 = 0$ ) and  $r_1 \neq 0$  (resp.  $q_1 \neq 0$ ) or  $q_1 \neq 0$  and  $r_1 \neq 0$ . The assertion for the first and the second case follows respectively by Proposition 2.2 and Corollary 2.4 of [9]. Now suppose that

$$\left( \sum_{i=1}^{s-1} I_{q_i} J_{r_i} \right)^* = I_{n-q_1+1} + \left( \sum_{i=1}^{s-2} I_{n-q_{i+1}+1} J_{m-r_i+1} \right) + J_{m-r_{s-1}+1}.$$

By equation 2.3 we have

$$(I_{q_s} J_{r_s} + \sum_{i=1}^{s-1} I_{q_i} J_{r_i})^* = (I_{q_s} J_{r_s})^* \cap \left( \sum_{i=1}^{s-1} I_{q_i} J_{r_i} \right)^*$$

that is equal to, by Corollary 2.4 of [9] and induction hypothesis,

$$(3.5) \quad (I_{n-q_s+1} + J_{m-r_s+1}) \cap (I_{n-q_1+1} + \sum_{i=1}^{s-2} I_{n-q_{i+1}+1} J_{m-r_i+1} + J_{m-r_{s-1}+1}).$$

We observe, since  $s > 1$  and  $q_s > q_i$  for all  $i$ , that  $q_s \neq 0$ . Let  $H = I_{n-q_1+1} + \sum_{i=1}^{s-2} I_{n-q_{i+1}+1} J_{m-r_i+1}$ . If we apply the modular law to 3.5 we have

$$(I_{n-q_s+1} + J_{m-r_s+1}) \cap H + (I_{n-q_s+1} + J_{m-r_s+1}) \cap J_{m-r_{s-1}+1}.$$

Since by hypothesis  $q_i < q_s \leq n$  we have  $I_{n-q_s+1} \supset H$  and observing that  $J_{m-r_s+1} \supset J_{m-r_{s-1}+1}$ , the assertion follows easily.  $\square$

**Remark 3.2.** By Proposition 3.1 we have that the class of mixed product ideals with a finite set of summand is closed under Alexander duality (see also [9], Remark 2.5).

**Corollary 3.3.** Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let

$$\mathcal{X}_i = \{X \subset \{x_1, \dots, x_n\} : |X| = i\}, \mathcal{Y}_j = \{Y \subset \{y_1, \dots, y_m\} : |Y| = j\},$$

with  $\mathcal{X}_i = \emptyset$  if  $i > n$  and  $\mathcal{Y}_j = \emptyset$  if  $j > m$ . Then

$$\sum_{i=1}^s I_{q_i} J_{r_i} = \mathcal{P}_x \cap \mathcal{P}_{xy} \cap \mathcal{P}_y$$

where

$$\begin{aligned} \mathcal{P}_x &= \bigcap_{X \in \mathcal{X}_{n-q_1+1}} (X), & \mathcal{P}_y &= \bigcap_{Y \in \mathcal{Y}_{m-r_s+1}} (Y), \\ \mathcal{P}_{xy} &= \bigcap_{i=1}^{s-1} \left( \bigcap_{X,Y} ((X) + (Y)) \right), & X &\in \mathcal{X}_{n-q_{i+1}+1}, Y \in \mathcal{Y}_{m-r_i+1}. \end{aligned}$$

*Proof.* By Alexander duality and Proposition 3.1 we have that

$$\sum_{i=1}^s I_{q_i} J_{r_i} = (I_{n-q_1+1} + \sum_{i=1}^{s-1} I_{n-q_{i+1}+1} J_{m-r_i+1} + J_{m-r_s+1})^*.$$

By equation 2.2 the assertion follows.  $\square$

**Corollary 3.4.** Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let  $\sum_{i=1}^s I_{q_i} J_{r_i}$  be the mixed product ideal on the ring  $S$ . Let  $h = \text{height} \sum_{i=1}^s I_{q_i} J_{r_i}$ , then the ideal is unmixed if and only if the following conditions are satisfied:

- (1)  $m + n - (q_{i+1} + r_i) + 2 = h, \forall i = 1, \dots, s-1$ ;
- (2) if  $q_1 > 0$  then  $n - q_1 + 1 = h$ ;
- (3) if  $r_s > 0$  then  $m - r_s + 1 = h$ .

**Definition 3.5.** Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let  $\sum_{i=1}^s I_{q_i} J_{r_i}$  be the mixed product ideal on the ring  $S$ . We define  $s' \in \mathbb{N}$  such that

$$s' = \begin{cases} s+1 & \text{if } q_1 > 0 \text{ and } r_s > 0 \\ s & \text{if } q_1 > 0 \text{ and } r_s = 0 \text{ or } q_1 = 0 \text{ and } r_s > 0 \\ s-1 & \text{if } q_1 = r_s = 0 \end{cases}$$

and the two vectors  $\bar{q} = (q(1), \dots, q(s'))$ ,  $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$  such that

$$\begin{aligned} q(i) &= \begin{cases} q_i - 1 & \text{if } q_1 > 0 \\ q_{i+1} - 1 & \text{if } q_1 = 0 \end{cases} \\ r(i) &= \begin{cases} r_{i-1} - 1 & \text{if } q_1 > 0 \\ r_i - 1 & \text{if } q_1 = 0 \end{cases} \end{aligned}$$

with  $i = 1, \dots, s'$  and  $r_0 = m + 1$  and  $q_{s+1} = n + 1$ .

**Proposition 3.6.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal. Using the notation of Definition 3.5 there exists a partition of  $\mathcal{F}(\Delta)$ ,  $\mathcal{F}(\Delta) = \mathcal{F}_1(\Delta) \cup \dots \cup \mathcal{F}_{s'}(\Delta)$  such that*

$$(3.6) \quad \mathcal{F}_k(\Delta) = \{ \{x_{i_1}, \dots, x_{i_{q(k)}}, y_{j_1}, \dots, y_{j_{r(k)}}\} : \\ 1 \leq i_1 < \dots < i_{q(k)} \leq n, 1 \leq j_1 < \dots < j_{r(k)} \leq m \},$$

with  $k = 1, \dots, s'$ .

*Proof.* By Corollary 3.3 and equation 2.1 the assertion follows.  $\square$

From now on we associate to a mixed product ideal  $\sum_{i=1}^s I_{q_i} J_{r_i}$  the value  $s' \in \mathbb{N}$  and the vectors  $\bar{q} = (q(1), \dots, q(s'))$ ,  $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$  defined in 3.5. We also give, for the sake of completeness, a way to compute the sequences  $0 \leq q_1 < \dots < q_s \leq n$ ,  $0 \leq r_s < \dots < r_1 \leq m$  by the vectors  $\bar{q} = (q(1), \dots, q(s'))$  and  $\bar{r} = (r(1), \dots, r(s'))$ .

**Definition 3.7.** *Let  $s' \in \mathbb{N}$ ,  $\bar{q} = (q(1), \dots, q(s'))$ ,  $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$ , with  $0 \leq q(1) < \dots < q(s') \leq n$ ,  $0 \leq r(s') < \dots < r(1) \leq m$ . We define  $s \in \mathbb{N}$  such that*

$$s = \begin{cases} s' - 1 & \text{if } r(1) = m \text{ and } q(s') = n \\ s' & \text{if } r(1) = m \text{ and } q(s') < n \text{ or } r(1) < m \text{ and } q(s') = n \\ s' + 1 & \text{if } r(1) < m \text{ and } q(s') < n \end{cases}$$

and the two sequences  $0 \leq q_1 < \dots < q_s \leq n$ ,  $0 \leq r_s < \dots < r_1 \leq m$  such that

$$q_i = \begin{cases} q(i) + 1 & \text{if } r(1) = m \\ q(i-1) + 1 & \text{if } r(1) < m \end{cases} \\ r_i = \begin{cases} r(i+1) + 1 & \text{if } r(1) = m \\ r(i) + 1 & \text{if } r(1) < m \end{cases}$$

with  $i = 1, \dots, s$  and  $q(0) = r(s' + 1) = -1$ .

**Lemma 3.8.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ ,  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal and keep the notation of Proposition 3.6. Then for each  $F \in \mathcal{F}_i(\Delta)$  and for each  $G \in \mathcal{F}_j(\Delta)$  with  $1 \leq i < j \leq s'$  we have*

$$\dim F \cap G \leq q(i) + r(j) - 1.$$

*Proof.* By Proposition 3.6 we have  $|F| = q(i) + r(i)$  and  $|G| = q(j) + r(j)$ . By the ordering in 3.2 and 3.3 and the Definition 3.5 we have  $q(i) < q(j)$ ,  $r(i) > r(j)$  for all  $1 \leq i < j \leq s'$  and the assertion follows.  $\square$

**Lemma 3.9.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal and keep the notation of Proposition 3.6. Let  $I_\Delta$  be unmixed and let  $F \in \mathcal{F}_i(\Delta)$  and  $G \in \mathcal{F}_j(\Delta)$  with  $\dim F \cap G = \dim \Delta - 1$ . If  $i < j$  (resp.  $i > j$ ) then*

$$(1) \quad j = i + 1 \text{ (resp. } j = i - 1);$$

- (2)  $q(i+1) = q(i) + 1$  (resp.  $q(i-1) = q(i) - 1$ );
- (3)  $r(i+1) = r(i) - 1$  (resp.  $r(i-1) = r(i) + 1$ ).

*Proof.* (1) We assume  $i < j$ . By Lemma 3.8 and since  $\Delta$  is pure we have the following inequality

$$(3.7) \quad \dim F \cap G = \dim \Delta - 1 = q(i) + r(i) - 2 \leq q(i) + r(j) - 1.$$

Since  $r(j) > r(i)$  and by the inequality 3.7 we obtain  $r(j) < r(i) \leq r(j) + 1$  that is  $r(i) = r(j) + 1$ . Therefore  $j = i + 1$  and  $r(i + 1) = r(i) - 1$ . By similar arguments we easily complete the proof of the assertion.  $\square$

**Lemma 3.10.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal. If  $q(i+1) = q(i) + 1$  for  $i = 1, \dots, s' - 1$  then  $\Delta$  shellable.*

*Proof.* We consider the partition  $\mathcal{F}(\Delta) = \mathcal{F}_1(\Delta) \cup \dots \cup \mathcal{F}_{s'}(\Delta)$  defined in Proposition 3.6. We set a linear order  $\prec$  on the facets  $\mathcal{F}(\Delta)$  such that  $F \prec G$  with  $F \in \mathcal{F}_k(\Delta)$ ,  $G \in \mathcal{F}_{k'}(\Delta)$  if either  $k < k'$  or  $k = k'$  with

$$F = \{x_{i_1}, \dots, x_{i_{q(k)}}, y_{j_1}, \dots, y_{j_{r(k)}}\}, G = \{x_{i'_1}, \dots, x_{i'_{q(k)}}, y_{j'_1}, \dots, y_{j'_{r(k)}}\},$$

$1 \leq i_1 < \dots < i_{q(k)} \leq n$ ,  $1 \leq j_1 < \dots < j_{r(k)} \leq m$ ,  $1 \leq i'_1 < \dots < i'_{q(k)} \leq n$ ,  $1 \leq j'_1 < \dots < j'_{r(k)} \leq m$  and there exists  $p$ ,  $1 \leq p \leq q(k)$ , such that  $i_k = i'_k$  for  $k = 1, \dots, p-1$  but  $i_p < i'_p$  or  $i_k = i'_k$  for all  $k = 1, \dots, q(k)$  and exists  $p'$ ,  $1 \leq p' \leq r(k)$ , such that  $j_k = j'_k$  for  $k = 1, \dots, p'-1$  but  $j_{p'} < j'_{p'}$ .

Suppose  $F \prec G$  with  $F \in \mathcal{F}_i(\Delta)$  and  $G \in \mathcal{F}_j(\Delta)$  with  $i < j$ . Since  $q(i) < q(j)$  there exists  $x_k \in G \setminus F$ . Now let  $G_k = G \setminus \{x_k\}$ . We observe that there exists  $F_k \in \mathcal{F}_{j-1}(\Delta)$  such that  $F_k \supset G_k$ , in fact by hypothesis  $q(j-1) = q(j) - 1$  and  $r(j-1) > r(j)$ . Hence  $G \setminus F_k = \{x_k\}$ .

Suppose  $F \prec G$  with  $F, G \in \mathcal{F}_i(\Delta)$ . We may assume  $x_k \in G \setminus F$ , in fact if such  $x_k$  does not exist we can consider the case  $y_k \in G \setminus F$  in an analogous way. Since  $F \prec G$  there exists  $x_{k'} \in F \setminus G$  such that  $k' < k$ . We set  $F_k = (G \setminus \{x_k\}) \cup \{x_{k'}\}$ . We observe that  $F_k \in \mathcal{F}_i(\Delta)$ ,  $F_k \prec G$  and  $G \setminus F_k = \{x_k\}$ . The assertion follows.  $\square$

By the same argument we have the following

**Lemma 3.11.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$  and let  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal. If  $r(i+1) = r(i) - 1$  for  $i = 1, \dots, s' - 1$  then  $\Delta$  is shellable.*

**Theorem 3.12.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ ,  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal,  $K[\Delta] = S/I_\Delta$ . The following conditions are equivalent:*

- (1)  $q(i+1) = q(i) + 1$  and  $r(i+1) = r(i) - 1$  for  $i = 1, \dots, s' - 1$ ;
- (2)  $\Delta$  is pure shellable;
- (3)  $K[\Delta]$  is Cohen-Macaulay;
- (4)  $\Delta$  is strongly connected.

*Proof.* (1) $\Rightarrow$ (2). By Lemma 3.10 (or equivalently 3.11) we have that  $K[\Delta]$  is shellable. We observe that  $q(i+1)+r(i+1)=q(i)+r(i)$  for  $i=1, \dots, s'-1$ . Hence  $\Delta$  is pure.

(2) $\Rightarrow$ (3). Always true.

(3) $\Rightarrow$ (4). Always true.

(4) $\Rightarrow$ (1). Let  $i=1, \dots, s'-1$  and let  $F \in \mathcal{F}_i(\Delta)$  and  $G \in \mathcal{F}_{i+1}(\Delta)$ . Since  $\Delta$  is strongly connected there exists a sequence of facets  $F = F_0, F_1, \dots, F_t = G$  such that  $\dim F_k \cap F_{k+1} = \dim \Delta - 1$  for  $k=0, \dots, t-1$ . We observe that there exists  $k \in \{0, \dots, t-1\}$  such that

$$F_k \in \bigcup_{j \leq i} \mathcal{F}_j(\Delta), \quad F_{k+1} \in \bigcup_{j \geq i+1} \mathcal{F}_j(\Delta).$$

Let  $F_k \in \mathcal{F}_{i-d}(\Delta)$  and  $F_{k+1} \in \mathcal{F}_{i+1+d'}(\Delta)$  with  $0 \leq d \leq i-1$ ,  $0 \leq d' \leq s'-i-1$ . Since  $q(i-d) \leq q(i)-d$  and  $r(i+1+d') \leq r(i)-1-d'$ , by Lemma 3.8 we obtain

$$\dim F_k \cap F_{k+1} \leq q(i) + r(i) - (d + d') - 2.$$

On the other hand  $\dim F_k \cap F_{k+1} = \dim \Delta - 1 = q(i) + r(i) - 2$ . Hence  $d = d' = 0$ . The assertion follows by Lemma 3.9.  $\square$

#### 4. SEQUENTIALLY COHEN-MACAULAY MIXED PRODUCT IDEALS

In this section we classify the sequentially Cohen-Macaulay mixed product ideals. We recall some definitions and results useful for our purpose and we continue to use the notation defined in section 3.

**Definition 4.1.** Let  $K$  be a field,  $S = K[x_1, \dots, x_n]$  be a polynomial ring. A graded  $S$ -module is called sequentially Cohen-Macaulay (over  $K$ ), if there exists a finite filtration of graded  $S$ -modules

$$0 = M_0 \subset M_1 \subset \dots \subset M_t = M$$

such that each  $M_i/M_{i-1}$  is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \dots < \dim(M_t/M_{t-1}).$$

**Definition 4.2.** Let  $\Delta$  be a simplicial complex then we define the pure simplicial complexes  $\Delta^{[l-1]}$  whose facets are

$$\mathcal{F}(\Delta^{[l-1]}) = \{F \in \Delta : \dim(F) = l-1\}, \quad 0 \leq l \leq \dim(\Delta) + 1.$$

A fundamental result about sequentially Cohen-Macaulay Stanley-Reisner rings  $K[\Delta]$  is the following

**Theorem 4.3** ([2]).  $K[\Delta]$  is sequentially Cohen-Macaulay if and only if  $K[\Delta^{[l-1]}]$  is Cohen-Macaulay for  $0 \leq l \leq \dim(\Delta) + 1$ .

**Remark 4.4.** Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ ,  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal,  $K[\Delta] = S/I_\Delta$  and let  $\mathcal{F}(\Delta)$  be partitioned as shown in Proposition 3.6, that is  $\mathcal{F}(\Delta) = \mathcal{F}_1(\Delta) \cup \dots \cup \mathcal{F}_{s'}(\Delta)$  such that

$$\mathcal{F}_k(\Delta) = \{\{x_{i_1}, \dots, x_{i_{q(k)}}, y_{j_1}, \dots, y_{j_{r(k)}}\} : \\ 1 \leq i_1 < \dots < i_{q(k)} \leq n, 1 \leq j_1 < \dots < j_{r(k)} \leq m\},$$

with  $k = 1, \dots, s'$ . If we set an  $l$  with  $0 \leq l \leq \dim(\Delta) + 1$  then for each  $k \in \{1, \dots, s'\}$  we have that  $\mathcal{F}_k(\Delta^{[l-1]}) = \mathcal{F}_{k1} \cup \dots \cup \mathcal{F}_{kt_k}$  where

$$\mathcal{F}_{kj} = \{\{x_{i_1}, \dots, x_{i_{q_k(j)}}, y_{j_1}, \dots, y_{j_{r_k(j)}}\} : \\ 1 \leq i_1 < \dots < i_{q_k(j)} \leq n, 1 \leq j_1 < \dots < j_{r_k(j)} \leq m\} \text{ with } j = 1, \dots, t_k,$$

satisfies the following properties:

- (1)  $q_k(t_k) = \min\{q(k), l\}$ ,
- (2)  $r_k(1) = \min\{r(k), l\}$ ,
- (3)  $q_k(i) = q_k(i+1) - 1$ ,  $r_k(i+1) = r_k(i) - 1$  for  $i = 1, \dots, t_k - 1$ .

**Definition 4.5.** Let  $\bar{q} = (q(1), \dots, q(s'))$ ,  $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$ , we define the following function  $\sigma : \{1, \dots, s'\} \rightarrow \mathbb{Z}_{\geq 0}$

$$\sigma(i) = q(i) + r(i).$$

**Lemma 4.6.** Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ ,  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal and let  $K[\Delta] = S/I_\Delta$ . If  $K[\Delta]$  is sequentially Cohen-Macaulay then

- (1) for all  $i \in \{1, \dots, s' - 1\}$  either  $q(i) = q(i+1) - 1$  or  $r(i) = r(i+1) + 1$ ;
- (2) there exists  $k \in \{1, \dots, s'\}$  such that  $\sigma(1) \leq \sigma(2) \leq \dots \leq \sigma(k) \geq \sigma(k+1) \geq \sigma(s')$ .

*Proof.* If  $K[\Delta]$  is sequentially Cohen-Macaulay then  $K[\Delta^{[l-1]}]$  is Cohen-Macaulay for  $0 \leq l \leq \dim \Delta + 1$ . Hence  $\Delta^{[l-1]}$  is strongly connected for  $0 \leq l \leq \dim \Delta + 1$ . We observe that if we negate property (1) (resp. (2)) we find an  $l$  such that  $\Delta^{[l-1]}$  is not strongly connected.

(1) We suppose that there exists  $k$  with  $1 \leq k \leq s' - 1$  such that

$$q(k) < q(k+1) - 1 \text{ and } r(k) > r(k+1) + 1.$$

Let  $l = \min\{\sigma(k), \sigma(k+1)\}$  and we assume that  $l = \sigma(k)$ . We observe that

$$(4.1) \quad \mathcal{F}(\Delta^{[l-1]}) = \bigcup_{i=1}^{s'} \mathcal{F}_i(\Delta^{[l-1]})$$

where the union is not disjoint. Since  $l = \sigma(k) \leq \sigma(k+1)$  we have  $\mathcal{F}_k(\Delta^{[l-1]}) = \mathcal{F}_k(\Delta)$  and  $\mathcal{F}_{k+1}(\Delta^{[l-1]}) \neq \emptyset$ .

We show that for all  $F \in \mathcal{F}_i(\Delta)$  with  $\mathcal{F}_i(\Delta^{[l-1]}) \neq \emptyset$  and for all  $G \in \mathcal{F}_j(\Delta)$  with  $\mathcal{F}_j(\Delta^{[l-1]}) \neq \emptyset$  with  $1 \leq i \leq k < j \leq s'$  we have that

$$(4.2) \quad \dim F \cap G < \dim \Delta^{[l-1]} - 1.$$



If the inequality 4.2 is satisfied also the facets  $F' \subset F$  with  $F' \in \mathcal{F}_i(\Delta^{[l-1]})$  and  $G' \subset G$  with  $G' \in \mathcal{F}_j(\Delta^{[l-1]})$  inherit this property.

Hence  $\Delta^{[l-1]}$  is not strongly connected.

By Lemma 3.8 we have that  $\dim F \cap G \leq q(i) + r(j) - 1 \leq q(k) + r(j) - 1$ . Since  $r(k) > r(k+1) + 1$  we have  $q(k) + r(k) > q(k) + r(k+1) + 1$  that is

$$l - 2 = q(k) + r(k) - 2 > q(k) + r(j) - 1 \geq \dim F \cap G.$$

The case  $l = \sigma(k+1)$  follows by similar arguments.

(2) We suppose that there exist  $k, k^-, k^+$ , with  $1 \leq k^- < k < k^+ \leq s'$  such that

$$\sigma(k^-) > \sigma(k) < \sigma(k^+).$$

Let  $l = \min\{\sigma(k^-), \sigma(k^+)\}$  and we assume that  $l = \sigma(k^-)$ .

Hence  $\mathcal{F}_{k^-}(\Delta^{[l-1]}) = \mathcal{F}_{k^-}(\Delta)$  and, since  $l \leq \sigma(k^+)$ ,  $\mathcal{F}_{k^+}(\Delta^{[l-1]}) \neq \emptyset$ . By Lemma 3.8 and using similar arguments of (1) it is easy to show that, for all  $F \in \mathcal{F}_i(\Delta)$  with  $\mathcal{F}_i(\Delta^{[l-1]}) \neq \emptyset$  and for all  $G \in \mathcal{F}_j(\Delta)$  with  $\mathcal{F}_j(\Delta^{[l-1]}) \neq \emptyset$  with  $1 \leq i < k < j \leq s'$ ,

$$\dim F \cap G \leq q(i) + r(j) - 1 < q(k) + r(k) - 1 < l - 1.$$

The assertion follows since  $\mathcal{F}_k(\Delta^{[l-1]}) = \emptyset$ .  $\square$

We come to the main result of this section.

**Theorem 4.7.** *Let  $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ ,  $I_\Delta = \sum_{i=1}^s I_{q_i} J_{r_i}$  be a mixed product ideal and let  $K[\Delta] = S/I_\Delta$ . The following conditions are equivalent:*

- (1)  $K[\Delta]$  is sequentially Cohen-Macaulay.
- (2) The following conditions hold:
  - (a)  $q(i) = q(i+1) - 1$  or  $r(i) = r(i+1) + 1$  with  $i = 1, \dots, s' - 1$ ;
  - (b) there exists  $k \in \{1, \dots, s'\}$  such that  $\sigma(1) \leq \sigma(2) \leq \dots \leq \sigma(k) \geq \sigma(k+1) \geq \sigma(s')$ .

*Proof.* (1) $\Rightarrow$ (2). See Lemma 4.6.

(2) $\Rightarrow$ (1). We need to show that for all  $l$  with  $0 \leq l \leq \dim \Delta + 1$  we have  $K[\Delta^{[l-1]}]$  is Cohen-Macaulay. Let  $\Delta' = \Delta^{[l-1]}$ , by Remark 4.4 we have that

$$(4.3) \quad \mathcal{F}(\Delta') = \bigcup \mathcal{F}_{kj} \text{ with } k = 1, \dots, s', j = 1, \dots, t_k,$$

where

$$\mathcal{F}_{kj} \cap \mathcal{F}_{k'j'} = \begin{cases} \mathcal{F}_{kj} = \mathcal{F}_{k'j'} & \text{if } q_k(j) = q_{k'}(j') \\ \emptyset & \text{if } q_k(j) \neq q_{k'}(j') \end{cases}$$

for all  $k, k' \in \{1, \dots, s'\}$ ,  $j = 1, \dots, t_k$ ,  $j' = 1, \dots, t_{k'}$ . If we remove the redundant elements in 4.3 and sort the remaining ones in an increasing order by  $q_k(j)$  with  $k = 1, \dots, s'$  and  $j = 1, \dots, t_k$ , we obtain a partition, with  $\bar{q}' = (q'(1), \dots, q'(t'))$ ,  $\bar{r}' = (r'(1), \dots, r'(t'))$  and  $q'(i) < q'(i+1)$  for  $i = 1, \dots, t' - 1$ . Since  $\Delta'$  is pure by definition, it is sufficient to show that  $q'(i+1) = q'(i) + 1$  for  $i = 1, \dots, t' - 1$  by Theorem 3.12.

Let  $q'(i)$  be an entry of the vector  $\vec{q}'$  with  $i = 1, \dots, t' - 1$ , then  $q'(i) < l$  and there exists  $q_k(j)$  related to 4.3 with  $q'(i) = q_k(j)$  with  $k = 1, \dots, s'$  and  $j = 1, \dots, t_k$ .

If  $j < t_k$  by property (3) of Remark 4.4 we are done. If  $j = t_k$  this implies that in the partition induced by 4.3 there exists  $k' > k$  such that  $\mathcal{F}_{k'}(\Delta') \neq \emptyset$ . Hence by the condition (1.b),  $\sigma(k+1) \geq \min\{\sigma(k), \sigma(k')\} \geq l$ , therefore  $\mathcal{F}_{k+1}(\Delta') \neq \emptyset$ . By condition (1.a), if  $q(k+1) = q(k) + 1$  and by property (1) of Remark 4.4 we have  $q_{k+1}(t_{k+1}) = q(k) + 1$ . If  $q(k+1) \neq q(k) + 1$  then  $r(k+1) = r(k) - 1$  and this implies by property (2) of Remark 4.4 that  $r_{k+1}(1) = r(k) - 1$ , hence  $q_{k+1}(1) = l - (r(k) - 1) = q(k) + 1$ . □

## REFERENCES

- [1] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge Univ. Press, Cambridge, 1997.
- [2] A. M. Duval, *Algebraic shifting and sequentially Cohen-Macaulay simplicial complexes* Electron. J. Combin. 3 (1996), n.1, Research Paper 21.
- [3] Faridi, S., *Simplicial trees are sequentially Cohen-Macaulay* J. Pure Appl. Algebra. **190** (2004), 121–136.
- [4] J. Herzog, T. Hibi, *Componentwise linear ideals*, Nagoya Math. J., **153** (1999), 141–153.
- [5] L. T. Hoa, N. D. Tam, *On some invariants of a mixed product of ideals*, Arch. Math. **94** (2010), 327–337.
- [6] C. Ionescu, G. Rinaldo, *Some algebraic invariants related to mixed product ideals*, Arch. Math. **91** (2008), 20–30.
- [7] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Springer-Verlag, Berlin, 2005.
- [8] G. Restuccia, R. Villarreal, *On the normality of monomial ideals of mixed products*, Commun. Algebra, **29** (2001), 3571–3580.
- [9] G. Rinaldo, *Betti numbers of mixed product ideals*, Arch. Math. **91** (2008), 416–426.
- [10] R. P. Stanley, *Combinatorics and Commutative Algebra, Second Edition*, Birkhäuser, Boston/Basel/Stuttgart, 1996.
- [11] A. Van Tuyl, R. Villarreal, *Shellable graphs and sequentially Cohen-Macaulay bipartite graphs*, Journ. of Combinat. Th. Ser. A, **115**, (2008), 799–814.
- [12] R. Villarreal, *Monomial algebras*, Marcel Dekker, New-York, 2001.

GIANCARLO RINALDO, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI MESSINA,  
VIALE FERDINANDO STAGNO D'ALCONTRES, 31, 98166 MESSINA. ITALY  
E-mail address: giancarlo.rinaldo@tiscali.it